Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 17, 2002*
Accepted: *November 7, 2002*

Neutrino wave packets in quantum field theory

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Abstract: We present a model of neutrino oscillations in the framework of quantum field theory in which the propagating neutrino and the particles participating to the production and detection processes are described by wave packets. The neutrino state is a superposition of massive neutrino wave packets determined by the production process, as naturally expected from causality. We show that the energies and momenta of the massive neutrino components relevant for neutrino oscillations are in general different from the average energies and momenta of the propagating massive neutrino wave packets, because of the effects of the detection process. Our results confirm the correctness of the standard expression for the oscillation length of extremely relativistic neutrinos and the existence of a coherence length.

Keywords: Solar and Atmospheric Neutrinos, Neutrino Physics.

http://jhep.sissa.it/archive/papers/jhep112002017/jhep112002017.pdf

JHEP11(2002)017

Contents

1. Introduction

The theory of neutrino oscillations is an important field of research, especially after the convincing evidence of the existence of neutrino oscillations obtained in atmospheric and solar neutrino experiments (see [1]–[7]).

Neutrinos can oscillate if neutrino mixing is realized in nature (see [8]–[11]), i.e. if the left-handed components $\nu_{\alpha L}$ of the flavor neutrino fields ($\alpha = e, \mu, \tau$) are superpositions of the left-handed components ν_{aL} of the fields of neutrinos with definite mass:

$$\nu_{\alpha L} = \sum_a U_{\alpha a} \nu_{aL} \,, \tag{1.1}$$

where U is the neutrino mixing matrix. Here the field ν_a describes a neutrino with mass m_a. A flavor neutrino produced in a charged-current weak interaction process is a superposition of massive neutrino components with well defined kinematical properties. Neutrino oscillations are space-dependent and/or time-dependent flavor transitions generated by the different phase velocities of different massive neutrino components. The probabilities of flavor transitions in neutrino oscillation experiments depend on the differences $\Delta m^2_{ab} \equiv m^2_a - m^2_b$ of the squared neutrino masses and on the elements of the mixing matrix U.

Different models of neutrino oscillations have been presented by several authors (see [10]–[14] and references therein). In this paper we adopt a wave packet approach, in the line of refs. [15]–[18].

As discussed in ref. [19], neutrino oscillations are possible only if the processes of neutrino production and detection have momentum uncertainties that allow the coherent production and detection of different massive neutrino components. From the fundamental principles of quantum mechanics it follows that the production and detection processes must be localized and the massive neutrino components of a flavor neutrino must be described by wave packets. We have developed a quantum-mechanical wave-packet description of neutrino oscillations in refs. [15, 16]. This model provides important insights in the physics of neutrino oscillations. The main one is the confirmation of the correctness of the standard expression for the oscillation phase of extremely relativistic neutrinos (see [8, 20, 21, 22]). In addition, the quantum-mechanical model of neutrino oscillations implies the existence of a coherence length, which was predicted in ref. [23] on the basis of intuitive quantum-mechanical arguments. For source–detector distances larger than the coherence length the flavor transition probability does not oscillate, because the separation of the different massive neutrino wave packets, which propagate with different group velocities, is so large that they cannot be absorbed coherently in the detection process [24]. In this case the probability of flavor-changing transitions is constant and depends only on the elements of the mixing matrix U.

In the quantum-mechanical model of neutrino oscillations the expression of the state describing a flavor neutrino as a superposition of massive neutrino components has to be assumed, because quantum mechanics is not sufficient for the description of the neutrino production and detection processes. Therefore, it is necessary to develop a description of neutrino oscillations in the framework of quantum field theory, which allows to calculate the amplitudes of neutrino production and detection. Moreover, one must notice that the theoretical prediction of neutrino oscillations follows from the mixing of neutrino fields in eq. (1.1) and a consistent theory of neutrino mixing and oscillations can be formulated only in the framework of quantum field theory.

In refs. [17, 18] we have developed a quantum-field-theoretical model of neutrino oscillations in which the particles taking part to the neutrino production and detection processes are described by localized wave packets and the neutrino propagating between the production and detection processes is a virtual particle (see also refs. [25, 26] for similar approaches, ref. [27, 28, 29] for another quantum-field-theoretical model of neutrino oscillations with virtual neutrinos, and refs. [30, 31] for a different quantum-field-theoretical point of view). This model confirms the correctness of the standard expression for the oscillation phase of extremely relativistic neutrinos and the existence of a coherence length.

However, since in oscillation experiments neutrinos propagate over macroscopic or even astronomical distances, we think that it is unnatural to consider them as virtual particles, with undefined properties (see refs. [18, 13]). Since massive neutrinos propagate as free particles between production and detection, it should be possible to describe their superposition constituting the flavor neutrino created in the production process by an appropriate quantum-field-theoretical state, similar to the quantum-mechanical wave-packet state in ref. [16]. Causality demands that the neutrino state is determined by the production process.

In this paper we present a quantum-field-theoretical model of neutrino oscillations in which the neutrino propagating between the production and detection processes is described by a wave packet state determined by the production process. This state is derived in section 2, starting from a production process in which the other interacting particles are described by wave packets. In section 3 we calculate the amplitude of the detection process occurring at a space-time distance $(\vec{L}, T)$ from the production process, using the neutrino state obtained in section 2 and wave packets for the other interacting particles. In section 4 we calculate the probability of neutrino oscillations in space from the average over the unmeasured propagation time T of the detection probability and we discuss the effects of the detection process on neutrino oscillations. Conclusions are presented in section 5.

In the following we assume that the particles that participate to the production and detection processes can be described by appropriate wave packets. This is possible if their properties are determined. If the information about their properties is incomplete, as often occurs in practice, each particle must be described by a statistical operator, also known as "density matrix", constructed from an incoherent mixture of wave packets with definite properties. As a consequence, also the propagating neutrino must be described by a statistical operator constructed from an incoherent mixture of the pure states that we derive in section 2, and the oscillation probability is given by an appropriate average of the oscillation probability that we derive in section 4 over the unknown properties of the particles participating to the production and detection processes.

2. Production

The approach presented here is based on the fact that in quantum field theory the effect of interactions is described by the operator $\mathcal{S} - \mathbf{1}$, with

$$\mathcal{S} = T \exp\left(-i \int \mathrm{d}^4x \mathcal{H}_I(x)\right), \tag{2.1}$$

where $\mathcal{H}_I(x)$ is the interaction hamiltonian expressed in terms of the appropriate field operators. Given an asymptotic initial state $|i\rangle$ the asymptotic final state resulting from an interaction is given by

$$|f\rangle \propto (\mathcal{S} - \mathbf{1})|i\rangle \simeq -i \int \mathrm{d}^4x \mathcal{H}_I(x)|i\rangle \,, \tag{2.2}$$

where the last expression applies to first order in perturbation theory, which we adopt in the following, because we consider weak interaction processes. Hence, in quantum field theory it is possible to calculate with eq. (2.2) the final state resulting from any interaction and in particular the final state of the processes in which a neutrino is produced. Since in the majority of experiments neutrinos are produced in charged-current weak decays, we consider the production process

$$P_I \to P_F + \ell_\alpha^+ + \nu_\alpha \,, \tag{2.3}$$

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in which P_I is the decaying particle, P_F is a decay product (absent in two-body decays) and ℓ_α^+ is the final state charged lepton that determines the flavor of the produced neutrino ν_α. As a simple example of a possible production process of the type (2.3) we will consider the pion decay

$$\pi^+ \to \mu^+ + \nu_\mu \,, \tag{2.4}$$

where $P_I = \pi^+$, P_F is absent and $\ell_\alpha^+ = \mu^+$.

Let us consider the production process (2.3). The final state obtained with eq. (2.2) describes all the final particles of the process in an entangled way:

$$|\widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto -i \int \mathrm{d}^4x \mathcal{H}_I^P(x) |P_I\rangle \,, \tag{2.5}$$

where $|P_I\rangle$ is the state describing the initial particle P_I and $\mathcal{H}_I^P(x)$ is the hamiltonian describing the production process.

The entangled state $|\widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle$ can be disentangled by measuring the properties of the particles involved. In particular, in the study of neutrino oscillations, one is interested in the knowledge of the state $|\nu_\alpha\rangle$ describing the neutrino produced in a process of type (2.3). In order to disentangle the state $|\nu_\alpha\rangle$, it is necessary to measure the properties of the other particle in the final state, i.e. of P_F and ℓ_α^+. This measurement does not have to be done necessarily by a specific instrument, but could be done by the interactions of the anti-lepton ℓ_α^+ and of P_F with the surrounding medium. The measurement process causes a collapse of the entangled final state $|\widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle$ to a disentangled state $|P_F\rangle|\ell_\alpha^+\rangle|\nu_\alpha\rangle$, with the neutrino state $|\nu_\alpha\rangle$ given by

$$|\nu_\alpha\rangle \propto \left(\langle P_F|\langle \ell_\alpha^+|\right) |\widetilde{P}_F, \widetilde{\ell}_\alpha^+, \widetilde{\nu}_\alpha\rangle \propto \left(\langle P_F|\langle \ell_\alpha^+|\right) - i \int \mathrm{d}^4x \mathcal{H}_I^P(x) |P_I\rangle \,. \tag{2.6}$$

The effective interaction hamiltonian that describes the process (2.3) in the Standard Model is

$$\begin{aligned} \mathcal{H}_I^P(x) &= \frac{G_F}{\sqrt{2}} \bar{\nu}_\alpha(x) \gamma^\rho \left(1 - \gamma_5\right) \ell_\alpha(x) J_\rho^P(x) \\ &= \frac{G_F}{\sqrt{2}} \sum_a U_{\alpha a}^* \bar{\nu}_a(x) \gamma^\rho \left(1 - \gamma_5\right) \ell_\alpha(x) J_\rho^P(x) \,, \end{aligned} \tag{2.7}$$

where G_F is the Fermi constant and $J_\rho^P(x)$ is the weak charged current that describes the transition $P_I \to P_F$.

In order to simplify as much as possible our discussion, let us consider the process (2.3) with the particles P_I, P_F and ℓ_α^+ described by the wave-packet states

$$|\chi\rangle = \int \mathrm{d}^3p \psi_\chi \left(\vec{p}; \vec{p}_\chi, \sigma_{p\chi}\right) |\chi\left(\vec{p}, h_\chi\right)\rangle \,, \tag{2.8}$$

where $\chi = P_I, P_F, \ell_\alpha^+$, the momentum distributions are denoted by $\psi_\chi(\vec{p}; \vec{p}_\chi, \sigma_{p\chi})$, and h_χ are the helicities. We assume that the particles P_I, P_F and ℓ_α^+ are polarized. If the measurement process is not sufficient to determine the polarization of these particles, each of them must be described by a statistical operator (density matrix) constructed from an incoherent mixture of the pure states (2.8) with different helicities. Consequently, the propagating neutrino must be described by a statistical operator constructed from an incoherent mixture of the pure states (2.6) obtained with different helicities of the particles P_I, P_F and ℓ_α^+.

We consider the gaussian momentum distributions[1]

$$\psi_\chi\left(\vec{p};\vec{p}_\chi,\sigma_{p\chi}\right) = \left(2\pi\sigma_{p\chi}^2\right)^{-3/4} \exp\left[-\frac{\left(\vec{p}-\vec{p}_\chi\right)^2}{4\sigma_{p\chi}^2}\right], \tag{2.9}$$

where $\vec{p}_\chi$ and $\sigma_{p\chi}$ are, respectively, the average momentum and the momentum uncertainty of the wave packet of the particle χ. The corresponding wave functions in coordinate space are given by

$$\psi_\chi\left(\vec{x},t;\vec{p}_\chi,\sigma_{p\chi}\right) = \langle 0|\chi(x)|\chi\rangle \approx \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} \psi_\chi\left(\vec{p};\vec{p}_\chi,\sigma_{p\chi}\right) e^{-iE_\chi(\vec{p})t+i\vec{p}\vec{x}}, \tag{2.10}$$

where

$$E_\chi(\vec{p}) = \sqrt{\vec{p}^2+m_\chi^2} \tag{2.11}$$

is the energy corresponding to the momentum $\vec{p}$, and we have neglected for simplicity the spin degrees of freedom. The gaussian momentum distributions (2.9) are assumed to be sharply peaked around their average momentum, i.e. the condition $\sigma_{p\chi} \ll E_\chi^2(\vec{p}_\chi)/m_\chi$ is assumed to be satisfied. Under this condition, the energy $E_\chi(\vec{p})$ can be approximated by[2]

$$E_\chi(\vec{p}) \simeq E_\chi + \vec{v}_\chi\left(\vec{p}-\vec{p}_\chi\right), \tag{2.12}$$

where

$$E_\chi \equiv E_\chi\left(\vec{p}_\chi\right) = \sqrt{\vec{p}_\chi^2+m_\chi^2} \tag{2.13}$$

is the average energy (up to corrections of the order $\sigma_{p\chi}^2/E_\chi$, see eq. (2.17)) and

$$\vec{v}_\chi \equiv \left.\frac{\partial E_\chi}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_\chi} = \frac{\vec{p}_\chi}{E_\chi} \tag{2.14}$$

is the group velocity of the wave packet of the particle χ. Under this approximation the integration in eq. (2.10) is gaussian and leads to

$$\psi_\chi\left(\vec{x},t;\vec{p}_\chi,\sigma_{p\chi}\right) \simeq \left(2\pi\sigma_{x\chi}^2\right)^{-3/4} \exp\left[-iE_\chi t + i\vec{p}_\chi\cdot\vec{x} - \frac{\left(\vec{x}-\vec{v}_\chi t\right)^2}{4\sigma_{x\chi}^2}\right], \tag{2.15}$$

where $\sigma_{x\chi}$ defined by the relation

$$\sigma_{x\chi}\sigma_{p\chi} = \frac{1}{2} \tag{2.16}$$

[1] A gaussian momentum distribution is the most convenient one for the calculation of several integrations in the following. Other distributions which are sharply peaked around an average momentum $\vec{p}_\chi$ lead to the same results after their approximation with a gaussian in order to perform the integrals with a saddle-point approximation. Therefore, the gaussian momentum distributions can be taken as approximations of the real momentum distributions from the beginning.

[2] With this approximation we neglect the spreading of the wave packets.

is the spatial width of the wave packet. One can see that the wave functions overlap at the origin of space-time coordinates, where the neutrino production process takes place. Hence, the wave packet states (2.8) are appropriate for the description of the particles taking part to the localized production process (2.3).

Let us determine the energy uncertainty of the wave packet state (2.8) describing a localized particle χ. Developing $E_\chi(\vec{p})$ up to second order in the power series of $(\vec{p}-\vec{p}_\chi)$, one obtains the average energy[3]

$$\langle E\rangle_\chi = \langle\chi|\widehat{E}|\chi\rangle \simeq E_\chi + \left(3-\vec{v}_\chi^{\,2}\right)\frac{\sigma_{p\chi}^2}{2E_\chi}\,, \tag{2.17}$$

where $\widehat{E}$ is the energy operator. The average squared energy is given exactly by

$$\left\langle E^2\right\rangle_\chi = \langle\chi|\widehat{\vec{P}}^{\,2}|\chi\rangle + m_\chi^2 = E_\chi^2 + 3\sigma_{p\chi}^2\,, \tag{2.18}$$

where $\widehat{\vec{P}}$ is the momentum operator, leading to the squared energy uncertainty

$$\left\langle(\delta E)^2\right\rangle_\chi = \langle\chi|(\widehat{E}-E_\chi)^2|\chi\rangle = \vec{v}_\chi^{\,2}\sigma_{p\chi}^2\,. \tag{2.19}$$

Therefore, somewhat surprisingly, a localized particle at rest has momentum uncertainty without energy uncertainty (at order $\sigma_{p\chi}/E_\chi$).

We use the following Fourier expansion for the spin 1/2 fermion fields:

$$\mathfrak{f}(x) = \int\frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\sum_{h=\pm1}\left[a_\mathfrak{f}(\vec{p},h)u_\mathfrak{f}(\vec{p},h)e^{-ipx} + b_\mathfrak{f}^\dagger(\vec{p},h)v_\mathfrak{f}(\vec{p},h)e^{ipx}\right]_{p^0=E_\mathfrak{f}(\vec{p})}\,, \tag{2.20}$$

where h is the helicity, $u_\mathfrak{f}(\vec{p},h)$ and $v_\mathfrak{f}(\vec{p},h)$ are four-component spinors, and $a_\mathfrak{f}(\vec{p},h)$ and $b_\mathfrak{f}(\vec{p},h)$ are the particle and anti-particle destruction operators obeying the canonical anti-commutation relations

$$\left\{a_\mathfrak{f}(\vec{p},h),a_\mathfrak{f}^\dagger(\vec{p}\,',h')\right\} = \left\{b_\mathfrak{f}(\vec{p},h),b_\mathfrak{f}^\dagger(\vec{p}\,',h')\right\} = \delta^3(\vec{p}-\vec{p}\,')\delta_{hh'}\,. \tag{2.21}$$

The one-particle fermion states with definite momentum and helicity $|\mathfrak{f}(\vec{p},h)\rangle = a_\mathfrak{f}^\dagger(\vec{p},h)|0\rangle$ are normalized by the relation $\langle\mathfrak{f}(\vec{p},h)|\mathfrak{f}(\vec{p}\,',h')\rangle = \delta^3(\vec{p}-\vec{p}\,')\delta_{hh'}$. One can easily check that in this way the wave-packet states (2.8) for $\chi=\mathfrak{f}$ are normalized to $\langle\mathfrak{f}|\mathfrak{f}\rangle = 1$.

From eqs.(2.6), (2.7) and (2.8), the state that describes the neutrino produced in the process (2.3) is given by

$$\begin{aligned}
|\nu_\alpha\rangle \propto \sum_a U_{\alpha a}^*\int\mathrm{d}^3p'_{P_F}\psi_{P_F}^*\left(\vec{p}\,'_{P_F};\vec{p}_{P_F},\sigma_{pP_F}\right)\int\mathrm{d}^3p'_{\ell_\alpha^+}\psi_{\ell_\alpha^+}^*\left(\vec{p}\,'_{\ell_\alpha^+};\vec{p}_{\ell_\alpha^+},\sigma_{\ell_\alpha^+}\right)\times\\
\times\int\mathrm{d}^3p'_{P_I}\psi_{P_I}\left(\vec{p}\,'_{P_I};\vec{p}_{P_I},\sigma_{pP_I}\right)\times\\
\times\int\mathrm{d}^4x\left(\langle P_F(\vec{p}\,'_{P_F})|\langle\ell_\alpha^+(\vec{p}\,'_{\ell_\alpha^+})|\right)\bar{\nu}_a(x)\gamma^\rho\left(1-\gamma_5\right)\ell_\alpha(x)J_\rho^P(x)|P_I\left(\vec{p}\,'_{P_I}\right)\rangle\,.
\end{aligned} \tag{2.22}$$

[3] I would like to thank M. Beuthe for pointing out the necessity to develop $E_\chi(\vec{p})$ up to second order in the power series of $(\vec{p}-\vec{p}_\chi)$. As a consequence, eqs. (2.17), (2.19), (2.52), (2.53) and (2.70) have been corrected with respect to the first version of the paper appeared in the electronic archive hep-ph, where $E_\chi(\vec{p})$ was developed only to first order (as in eq. (2.12)).

Using the Fourier expansion (2.20) for the lepton fields, we obtain

$$
\begin{aligned}
|\nu_\alpha\rangle \propto & \sum_a U_{\alpha a}^* \int \mathrm{d}^4x \int \mathrm{d}^3p'_{P_F} \psi_{P_F}^*(\vec{p}\,'_{P_F};\vec{p}_{P_F},\sigma_{pP_F}) \times \int \mathrm{d}^3p'_{\ell_\alpha^+} \psi_{\ell_\alpha^+}^* \left(\vec{p}\,'_{\ell_\alpha^+};\vec{p}_{\ell_\alpha^+},\sigma_{\ell_\alpha^+}\right) \\
& \int \mathrm{d}^3p'_{P_I} \psi_{P_I}(\vec{p}\,'_{P_I};\vec{p}_{P_I},\sigma_{pP_I}) J_\rho^P(\vec{p}\,'_{P_F},h_{P_F};\vec{p}\,'_{P_I},h_{P_I}) \\
& \times \int \mathrm{d}^3p \sum_h \bar{u}_{\nu_a}(\vec{p},h)\gamma^\rho\,(1-\gamma_5)\, v_{\ell_\alpha^+}(\vec{p}\,'_{\ell_\alpha^+},h_{\ell_\alpha^+}) e^{i(p+p'_{\ell_\alpha^+}+p'_{P_F}-p'_{P_I})x} |\nu_a(\vec{p},h)\rangle\,,
\end{aligned}
\tag{2.23}
$$

with $p^0 = E_{\nu_a}(\vec{p})$, $p'^0_{P_I} = E_{P_I}(\vec{p}\,'_{P_I})$, $p'^0_{P_F} = E_{P_F}(\vec{p}\,'_{P_F})$, $p'^0_{\ell_\alpha^+} = E_{\ell_\alpha^+}(\vec{p}\,'_{\ell_\alpha^+})$, and $J_\rho^P(\vec{p}\,'_{P_F},h_{P_F};\vec{p}\,'_{P_I},h_{P_I}) = \langle P_F\,(\vec{p}\,'_{P_F},h_{P_F})\,|\,J_\rho^P(0)\,|\,P_I\,(\vec{p}\,'_{P_I},h_{P_I})\rangle$. For example, in the pion decay process (2.4) $J_\rho^P(\vec{p}\,'_{P_F},h_{P_F};\vec{p}\,'_{P_I},h_{P_I})$ is given by $\langle 0|J_\rho^P(0)|\pi^+(\vec{p}\,'_{\pi^+})\rangle = f_\pi(p'_{\pi^+})_\rho$, where f_π is the pion decay constant.

Since the wave packets of P_I, P_F and ℓ_α^+ are assumed to be sharply peaked around the respective average momenta, the integrations over $\mathrm{d}^3p'_{P_F}$, $\mathrm{d}^3p'_{\ell_\alpha^+}$ and $\mathrm{d}^3p'_{P_I}$ can be performed with a saddle-point approximation using the approximation (2.12), which leads to

$$
\begin{aligned}
|\nu_\alpha\rangle \propto & \sum_a U_{\alpha a}^* \int \mathrm{d}^3p \sum_h \mathcal{A}_a^P(\vec{p},h)|\nu_a(\vec{p},h)\rangle \times \\
& \times \int \mathrm{d}^4x \exp\left[-i\left(E_P - E_{\nu_a}(\vec{p})\right)t + i(\vec{p}_P-\vec{p})\vec{x} - \frac{\vec{x}^2 - 2\vec{v}_P\cdot\vec{x}t + \Sigma_P t^2}{4\sigma_{xP}^2}\right],
\end{aligned}
\tag{2.24}
$$

with

$$E_P \equiv E_{P_I} - E_{P_F} - E_{\ell_\alpha^+}\,, \tag{2.25}$$

$$\vec{p}_P \equiv \vec{p}_{P_I} - \vec{p}_{P_F} - \vec{p}_{\ell_\alpha^+}\,, \tag{2.26}$$

$$\frac{1}{\sigma_{xP}^2} \equiv \frac{1}{\sigma_{xP_I}^2} + \frac{1}{\sigma_{xP_F}^2} + \frac{1}{\sigma_{x\ell_\alpha^+}^2}\,, \tag{2.27}$$

$$\vec{v}_P \equiv \sigma_{xP}^2 \left(\frac{\vec{v}_{P_I}}{\sigma_{xP_I}^2} + \frac{\vec{v}_{P_F}}{\sigma_{xP_F}^2} + \frac{\vec{v}_{\ell_\alpha^+}}{\sigma_{x\ell_\alpha^+}^2}\right), \tag{2.28}$$

$$\Sigma_P \equiv \sigma_{xP}^2 \left(\frac{\vec{v}_{P_I}^{\,2}}{\sigma_{xP_I}^2} + \frac{\vec{v}_{P_F}^{\,2}}{\sigma_{xP_F}^2} + \frac{\vec{v}_{\ell_\alpha^+}^{\,2}}{\sigma_{x\ell_\alpha^+}^2}\right), \tag{2.29}$$

$$\mathcal{A}_a^P(\vec{p},h) \equiv \bar{u}_{\nu_a}(\vec{p},h)\gamma^\rho\,(1-\gamma_5)\, v_{\ell_\alpha^+}\left(\vec{p}_{\ell_\alpha^+},h_{\ell_\alpha^+}\right) J_\rho^P\left(\vec{p}\,'_{P_F},h_{P_F};\vec{p}\,'_{P_I},h_{P_I}\right). \tag{2.30}$$

As naturally expected, the overall spatial width σ_{xP} of the production process is dominated by the smallest among the spatial widths of P_I, P_F and ℓ_α^+. The quantities $|\vec{v}_P|$ and Σ_P are limited by

$$0 \le |\vec{v}_P| \le 1\,, \qquad 0 \le \Sigma_P \le 1\,. \tag{2.31}$$

Carrying out the gaussian integral over d^4x, we obtain the neutrino state

$$|\nu_\alpha\rangle = N_\alpha \sum_a U_{\alpha a}^* \int \mathrm{d}^3p e^{-S_a^P(\vec{p})} \sum_h \mathcal{A}_a^P(\vec{p},h)|\nu_a(\vec{p},h)\rangle\,, \tag{2.32}$$

where N_α is a normalization constant such that

$$\langle \nu_\alpha | \nu_\alpha \rangle = 1 \,, \tag{2.33}$$

and

$$S_a^P(\vec{p}) \equiv \frac{(\vec{p}_P - \vec{p})^2}{4\sigma_{pP}^2} + \frac{[E_P - E_{\nu_a}(\vec{p}) - (\vec{p}_P - \vec{p}) \cdot \vec{v}_P]^2}{4\sigma_{pP}^2 \lambda_P} \,. \tag{2.34}$$

Here

$$\lambda_P \equiv \Sigma_P - \vec{v}_P^2 \,, \tag{2.35}$$

such that

$$0 \leq \lambda_P \leq 1 \,, \tag{2.36}$$

and we have defined the momentum uncertainty σ_{pP} through the relation

$$\sigma_{xP} \sigma_{pP} = \frac{1}{2} \,. \tag{2.37}$$

The overall momentum uncertainty is dominated by the largest momentum uncertainty among P_I, P_F and ℓ_α^+:

$$\sigma_{pP}^2 = \sigma_{pP_I}^2 + \sigma_{pP_F}^2 + \sigma_{p\ell_\alpha^+}^2 \,. \tag{2.38}$$

The neutrino state (2.32) describes a neutrino produced in the weak interaction process (2.3) as a superposition of massive neutrino components. In section 3 we discuss the detection of this state and in section 4 we derive the corresponding transition probability. In the following part of this section we discuss some properties of the massive neutrino wave-packet states

$$|\nu_a\rangle = N_a \int \mathrm{d}^3 p \, e^{-S_a^P(\vec{p})} \sum_h \mathcal{A}_a^P(\vec{p}, h) |\nu_a(\vec{p}, h)\rangle \,, \tag{2.39}$$

with the normalization constant N_a such that

$$\langle \nu_a | \nu_a \rangle = 1 \,. \tag{2.40}$$

The massive neutrino wave-packet states (2.39) are the components of the state (2.32), which can be written as

$$|\nu_\alpha\rangle = N_\alpha \sum_a \frac{U_{\alpha a}^*}{N_a} |\nu_a\rangle \,, \tag{2.41}$$

and the normalization constant N_α is related to the normalization constants N_a by

$$N_\alpha = \left(\sum_a \frac{|U_{\alpha a}|^2}{N_a^2} \right)^{-1/2} \,. \tag{2.42}$$

The normalization condition (2.40) requires that

$$N_a^2 \sum_h \int \mathrm{d}^3 p |\mathcal{A}_a^P(\vec{p}, h)|^2 e^{-2 S_a^P(\vec{p})} = 1 \,. \tag{2.43}$$

The integration over d^3p can be done with a saddle point approximation around the stationary point of $S_a^P(\vec{p})$,

$$\left.\frac{\partial S_a^P(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_a} = 0\,. \tag{2.44}$$

The momentum $\vec{p}_a$ is given by

$$\vec{p}_P - \vec{p}_a + \frac{1}{\lambda_P}\left[E_P - E_a - (\vec{p}_P - \vec{p}_a)\cdot\vec{v}_P\right](\vec{v}_a - \vec{v}_P) = 0\,, \tag{2.45}$$

with

$$E_a \equiv E_{\nu_a}(\vec{p}_a) = \sqrt{\vec{p}_a^{\,2} + m_a^2}\,, \tag{2.46}$$

$$\vec{v}_a \equiv \left.\frac{\partial E_{\nu_a}(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{p}_a} = \frac{\vec{p}_a}{E_a}\,. \tag{2.47}$$

The saddle-point approximation of the integration over d^3p in eq. (2.43) leads to

$$N_a = \left(\frac{\mathrm{Det}\Lambda_a}{\pi^3}\right)^{1/4} \frac{e^{S_a^P(\vec{p}_a)}}{\sqrt{\sum_h |\mathcal{A}_a^P(\vec{p}_a,h)|^2}}\,, \tag{2.48}$$

with

$$\begin{aligned}\Lambda_a^{jk} &\equiv \left.\frac{\partial^2 S_a^P(\vec{p})}{\partial p^j \partial p^k}\right|_{\vec{p}=\vec{p}_a} \\ &= \frac{\delta^{jk}}{2\sigma_{pP}^2} + \frac{\left(v_P^j - v_a^j\right)\left(v_P^k - v_a^k\right)}{2\sigma_{pP}^2\lambda_P} - \frac{\left[(E_P - E_a) - (\vec{p}_P - \vec{p}_a)\cdot\vec{v}_P\right]}{2\sigma_{pP}^2\lambda_P}\frac{\delta^{jk} - v_a^j v_a^k}{E_a}\,.\end{aligned} \tag{2.49}$$

Using the same saddle-point approximation for the integration over d^3p, one can find that $\vec{p}_a$ is the average momentum of the state $|\nu_a\rangle$,

$$\langle\vec{p}\rangle_a = \langle\nu_a|\widehat{\vec{P}}|\nu_a\rangle = \vec{p}_a\,, \tag{2.50}$$

and the uncertainties of the three momentum components are given by

$$\langle(\delta p^k)^2\rangle_a = \langle\nu_a|(\widehat{P}^k - p_a^k)^2|\nu_a\rangle = \frac{1}{2}(\Lambda_a^{-1})^{kk}\,. \tag{2.51}$$

Developing $E_{\nu_a}(\vec{p})$ up to second order in the power series of $(\vec{p} - \vec{p}_a)$ we obtain the average energy

$$\langle E\rangle_a = \langle\nu_a|\widehat{E}|\nu_a\rangle \simeq E_a + \frac{\mathrm{Tr}(\Lambda_a^{-1}) - \sum_{j,k} v_a^j (\Lambda_a^{-1})^{jk} v_a^k}{4E_a}\,, \tag{2.52}$$

and the energy uncertainty

$$\langle(\delta E)^2\rangle_a = \langle\nu_a|(\widehat{E} - E_a)^2|\nu_a\rangle \simeq \frac{1}{2}\sum_{j,k} v_a^j (\Lambda_a^{-1})^{jk} v_a^k\,. \tag{2.53}$$

In order to get further insight in the properties of the massive neutrino wave-packet states (2.39), it is necessary to solve eq. (2.45) and determine the values of $\vec{p}_a$ and E_a. It is important to notice that the production of the state $|\nu_\alpha\rangle$ in eq. (2.32) is not suppressed only if

$$S_a^P(\vec{p}_a) \lesssim 1 \tag{2.54}$$

for all values of the index a. Together with eq. (2.45), this inequality constraints the possible values of $\vec{p}_a$, $\vec{p}_P$ and E_P.

Equation (2.45) implies that the massive neutrino momenta $\vec{p}_a$ must be aligned in the direction of $\vec{p}_P$:

$$\vec{p}_a = p_a \vec{\ell}\,, \tag{2.55}$$

with $\vec{p}_P = p_P \vec{\ell}$ and $|\vec{\ell}| = 1$. Hence, $\vec{\ell}$ is the unit vector in the direction of propagation of the neutrino.

We solve eq. (2.45) in the approximation of extremely relativistic neutrinos. This approximation is valid in practice because only neutrinos with energy larger than some fraction of MeV are detectable. Indeed, neutrinos are detected in:

1. Charged-current or neutral-current weak processes which have an energy threshold larger than some fraction of MeV. This is due to the fact that in a scattering process

$$\nu + A \to \sum_X X \tag{2.56}$$

with A at rest, the squared center-of-mass energy $s = 2E_\nu m_A + m_A^2$ (neglecting the neutrino mass) must be bigger than $(\sum_X m_X)^2$, leading to

$$E_\nu^{\text{th}} = \frac{(\sum_X m_X)^2}{2m_A} - \frac{m_A}{2}\,. \tag{2.57}$$

For example:

- $E_\nu^{\text{th}} \simeq 0.233$MeV for $\nu_e + {}^{71}\text{Ga} \to {}^{71}\text{Ge} + e^-$ in the GALLEX [32], SAGE [33] and GNO [34] solar neutrino experiments.
- $E_\nu^{\text{th}} \simeq 0.81$MeV for $\nu_e + {}^{37}\text{Cl} \to {}^{37}\text{Ar} + e^-$ in the Homestake [35] solar neutrino experiment.
- $E_\nu^{\text{th}} \simeq 1.8$MeV for $\bar{\nu}_e + p \to n + e^+$ in reactor neutrino experiments (for example Bugey [36] and CHOOZ [37]).
- $E_\nu^{\text{th}} \simeq 2.2$MeV in the neutral-current process $\nu + d \to p + n + \nu$ used in the SNO experiment to detect active solar neutrinos [6].
- $E_\nu^{\text{th}} \simeq 110$MeV for $\nu_\mu + n \to p + \mu^-$.
- $E_\nu^{\text{th}} \simeq m_\mu^2/2m_e \simeq 10.9$GeV for $\nu_\mu + e^- \to \nu_e + \mu^-$.

2. The elastic scattering process $\nu + e^- \to \nu + e^-$, whose cross section is proportional to the neutrino energy ($\sigma(E_\nu) \sim \sigma_0 E_\nu/m_e$, with $\sigma_0 \sim 10^{-44}\text{cm}^2$). Therefore, an energy threshold of some MeV's is needed in order to have a signal above the background. For example, $E_\nu^{\text{th}} \simeq 5\,$MeV in the Super-Kamiokande [38] solar neutrino experiment.

Although the direct experimental upper limits for the effective neutrino masses in lepton decays are not very stringent ($m_{\nu_e} \lesssim 3\text{eV}$, $m_{\nu_\mu} \lesssim 190\text{keV}$, $m_{\nu_\tau} \lesssim 18.2\text{MeV}$, see ref. [39]), we know that the sum of the masses of the neutrinos that have a substantial mixing with ν_e, ν_μ and ν_τ is constrained to be smaller than a few eV by their contribution to the total energy density of the Universe [40, 41, 42].

The comparison of the cosmological limit on neutrino masses with the energy threshold in the processes of neutrino detection implies that the main massive neutrino components of detectable flavor neutrinos are extremely relativistic.[4]

We write the average massive neutrino energies E_a in the relativistic approximation[5] as

$$E_a \simeq E + \xi \frac{m_a^2}{2E}, \tag{2.58}$$

where E is the neutrino energy in the limit of zero mass. The corresponding momentum has modulus

$$p_a \simeq E - (1-\xi)\,\frac{m_a^2}{2E}. \tag{2.59}$$

In the following we consider

$$\vec{p}_P = E_P \vec{\ell}\,, \tag{2.60}$$

which corresponds to exact energy and momentum conservation in the production process in the case of massless neutrinos. It is possible to consider deviations from eq. (2.60) compatible with the inequality (2.54), but such deviations would entail a considerable complication of the formalism without further insights in the physical properties of neutrinos emitted in the process (2.3).

At zeroth order in m_a^2/E^2, eq. (2.45) imply

$$E = E_P\,. \tag{2.61}$$

The solution of eq. (2.45) at first order in m_a^2/E^2 gives the value of ξ:

$$\xi = \frac{\lambda_P - \vec{\ell}\cdot\vec{v}_P\left(1-\vec{\ell}\cdot\vec{v}_P\right)}{\lambda_P + \left(1-\vec{\ell}\cdot\vec{v}_P\right)^2}\,. \tag{2.62}$$

[4]It is still possible that the three active flavor neutrinos ν_e, ν_μ, ν_τ have very small mixing with heavy massive neutrinos that are not relativistic in some experiment. In this case, the heavy neutrino masses must be taken into account in the calculation of the production and detection rates of the heavy massive neutrinos, but the oscillations generated by the large mass differences between light and heavy neutrinos are too fast to be observable. Therefore, in practice it is sufficient to consider an incoherent mixture of light and heavy massive neutrinos that generate a constant flavor-changing transition probability. Also the mass differences between possible heavy neutrinos are expected to be too large to generate observable oscillations. Therefore, in the following we study the oscillations due to the mixing of the flavor neutrinos ν_e, ν_μ and ν_τ with light extremely relativistic massive neutrinos.

[5] In principle it is possible to consider the approximation of almost degenerate but not relativistic neutrinos [26]. This approximation could be relevant in practice only if the three active flavor neutrinos ν_e, ν_μ and ν_τ mix with heavy and almost degenerate massive neutrinos, such that the small mass differences among heavy neutrinos generate observable oscillations. Since this possibility seems very unlikely, we do not consider the case of almost degenerate but not relativistic massive neutrinos. Notice, however, that a similar approximation is important in the analogous quantum-field-theoretical treatment of meson oscillations (see [13]).

Hence, the value of the parameter ξ that determines the average energy and momentum of the produced neutrino state depends on the characteristics of the production process through the quantities λ_P and $\vec{v}_P$.

Using eqs. (2.55), (2.60) and the relativistic approximations (2.58), (2.59), with ξ given by eq. (2.62), we find

$$S_a^P(\vec{p}_a) = \left(\frac{m_a^2}{4E\sigma_{pP}}\right)^2 \left[\lambda_P + \left(1 - \vec{\ell}\cdot\vec{v}_P\right)^2\right]^{-1} . \tag{2.63}$$

Hence, the condition (2.54) is satisfied if

$$m_a^2 \lesssim \sigma_{pP} E \sqrt{\lambda_P + \left(1 - \vec{\ell}\cdot\vec{v}_P\right)^2} \tag{2.64}$$

for all values of a. This inequality shows that a finite momentum uncertainty σ_{pP} of the production process is necessary in order to produce coherently different extremely relativistic massive neutrino components.

The momentum and energy uncertainties in eqs. (2.51) and (2.53) can be estimated at zeroth order in m_a^2/E^2 inverting the symmetric matrix

$$\Lambda = \frac{1}{2\sigma_{pP}^2} \begin{pmatrix} 1 + \frac{\left(v_P^x - 1\right)^2}{\lambda_P} & \frac{\left(v_P^x - 1\right) v_P^y}{\lambda_P} & \frac{\left(v_P^x - 1\right) v_P^z}{\lambda_P} \\ \frac{\left(v_P^x - 1\right) v_P^y}{\lambda_P} & 1 + \frac{(v_P^y)^2}{\lambda_P} & \frac{v_P^y v_P^z}{\lambda_P} \\ \frac{\left(v_P^x - 1\right) v_P^z}{\lambda_P} & \frac{v_P^y v_P^z}{\lambda_P} & 1 + \frac{(v_P^z)^2}{\lambda_P} \end{pmatrix} . \tag{2.65}$$

From the determinant

$$\mathrm{Det}\Lambda = \frac{1}{(2\sigma_{pP}^2)^3} \left[1 + \frac{(v_P^x - 1)^2 + (v_P^y)^2 + (v_P^z)^2}{\lambda_P}\right] , \tag{2.66}$$

we obtain[6]

$$\langle (\delta p^x)^2 \rangle_a \simeq \frac{1}{2} \left(\Lambda^{-1}\right)^{xx} = \sigma_{pP}^2 \frac{1 + \frac{(v_P^y)^2 + (v_P^z)^2}{\lambda_P}}{1 + \frac{(v_P^x - 1)^2 + (v_P^y)^2 + (v_P^z)^2}{\lambda_P}} , \tag{2.67}$$

$$\langle (\delta p^y)^2 \rangle_a \simeq \frac{1}{2} \left(\Lambda^{-1}\right)^{yy} = \sigma_{pP}^2 \frac{1 + \frac{(v_P^x - 1)^2 + (v_P^z)^2}{\lambda_P}}{1 + \frac{(v_P^x - 1)^2 + (v_P^y)^2 + (v_P^z)^2}{\lambda_P}} , \tag{2.68}$$

$$\langle (\delta p^z)^2 \rangle_a \simeq \frac{1}{2} \left(\Lambda^{-1}\right)^{zz} = \sigma_{pP}^2 \frac{1 + \frac{(v_P^x - 1)^2 + (v_P^y)^2}{\lambda_P}}{1 + \frac{(v_P^x - 1)^2 + (v_P^y)^2 + (v_P^z)^2}{\lambda_P}} , \tag{2.69}$$

$$\langle (\delta E)^2 \rangle_a \simeq \frac{1}{2} \left(\Lambda^{-1}\right)^{xx} = \sigma_{pP}^2 \frac{1 + \frac{(v_P^y)^2 + (v_P^z)^2}{\lambda_P}}{1 + \frac{(v_P^x - 1)^2 + (v_P^y)^2 + (v_P^z)^2}{\lambda_P}} . \tag{2.70}$$

[6] I would like to thank M. Beuthe for finding misprints in eqs. (2.68) and (2.69) in the first version of the paper appeared in the electronic archive hep-ph.

Thus, the momentum and energy uncertainties are of the order of σ_{pP}, the total momentum uncertainty in the production process, unless the factors in eqs. (2.67)–(2.70) depending on λ_P and $\vec{v}_P$ assume extreme values.

In order to illustrate the meaning of our results, in the following Subsections we consider as an example neutrino production in the pion decay process (2.4) at rest ($\vec{v}_{\pi^+} = 0$) in four cases. From eqs. (2.60) and (2.61), the energy and momentum of the neutrino and muon at zeroth order in m_a^2/E^2 are determined by energy-momentum conservation in the case of massless neutrinos:

$$E = p_{\mu^+} = \frac{m_{\pi^+}^2 - m_{\mu^+}^2}{2m_{\pi^+}} \simeq 29.8\text{MeV}\,, \qquad E_{\mu^+} = \frac{m_{\pi^+}^2 + m_{\mu^+}^2}{2m_{\pi^+}} \simeq 109.8\text{MeV}\,, \tag{2.71}$$

leading to the muon velocity

$$v_{\mu^+} = \frac{m_{\pi^+}^2 - m_{\mu^+}^2}{m_{\pi^+}^2 + m_{\mu^+}^2} \simeq 0.27\,. \tag{2.72}$$

Equation (2.19) implies that the localized pion at rest has no energy uncertainty (at order $\sigma_{p\pi^+}/m_{\pi^+}$).

2.1 Unlocalized production process

In the limit

$$\sigma_{p\pi^+} \to 0\,, \qquad \sigma_{p\mu^+} \to 0 \quad \Longrightarrow \quad \sigma_{pP} \to 0\,, \tag{2.73}$$

the particles taking part to the production process and the production process itself are not localized.

In this case the condition (2.64) for the coherent production of different extremely relativistic massive neutrino components is not satisfied. Furthermore, no deviation from eq. (2.60) can satisfy the inequality (2.54) for more than one value of a, because in the limit (2.73) $S_a^P(\vec{p}_a)$ becomes infinite, suppressing the production of ν_a unless

$$(\vec{p}_P - \vec{p}_a)^2 + \frac{1}{\lambda_P}\left[E_P - E_a - (\vec{p}_P - \vec{p}_a)\cdot\vec{v}_P\right]^2 = 0\,. \tag{2.74}$$

Taking into account the fact that λ_P is positive, eq. (2.74) can be satisfied only if both squares are zero, i.e. if $\vec{p}_P = \vec{p}_a$ and $E_P = E_a$ exactly. Since this constraint can be satisfied only for one value of the index a, only the corresponding massive neutrino is produced.

2.2 Unlocalized pion

If the momentum of the pion is determined with high accuracy,

$$\sigma_{p\pi^+} \to 0\,, \tag{2.75}$$

the pion is unlocalized,

$$\sigma_{x\pi^+} \to +\infty\,, \tag{2.76}$$

and we have

$$\sigma_{xP} = \sigma_{x\mu^+}\,, \tag{2.77}$$

which imply (taking into account $\vec{v}_{\pi^+} = 0$)

$$\vec{v}_P = \vec{v}_{\mu^+} = -\vec{\ell} v_{\mu^+} \,, \qquad \Sigma_P = v_{\mu^+}^2 \,, \qquad \lambda_P = 0 \,. \tag{2.78}$$

From eq. (2.62) we get

$$\xi = \frac{1}{2}\left(1 - \frac{m_{\pi^+}^2}{m_{\mu^+}^2}\right) \simeq 0.21 \,. \tag{2.79}$$

This is the value of ξ given by exact energy-momentum conservation in the production process in the case of pion decay at rest and different muon momenta for each massive neutrino.

The condition (2.64) in this case reads

$$m_a^2 \lesssim \sigma_{p\mu^+} E \left(1 + v_{\mu^+}\right) \sim \sigma_{p\mu^+} m_{\pi^+} \,. \tag{2.80}$$

If the muon propagates in normal matter, one can estimate $\sigma_{x\mu^+}$ to be given approximately by the inter-atomic distance, $\sigma_{x\mu^+} \sim 10^{-8}\,\text{cm}$, which corresponds to $\sigma_{p\mu^+} \sim 10^3\,\text{eV}$. Thus, the condition (2.80) numerically gives $m_a^2 \lesssim 10^{11}\,\text{eV}^2$, that is certainly satisfied.

The production process has a momentum uncertainty given by eq. (2.77), and an energy uncertainty $v_{\mu^+}\sigma_{pP}$ due to the muon. From Eqs. (2.67) and (2.70) one can see that in this case the massive neutrino components of the neutrino state (2.32) have no energy uncertainty and no momentum uncertainty along the direction $\vec{\ell}$ of propagation. Hence, in this limiting case the energy and momentum uncertainties of the massive neutrino components of the neutrino state are rather different from the energy and momentum uncertainties of the production process. Only the uncertainties of the components of the momentum orthogonal to the direction of propagation, given by eqs. (2.68) and (2.69), are equal to the momentum uncertainty σ_{pP} of the production process. These uncertainties are actually necessary in order to localize the massive neutrino components along the direction of propagation.

2.3 Equal energy limit

In the limit in which the pion is localized but the final lepton is not localized,

$$\sigma_{x\mu^+} \to +\infty \,, \tag{2.81}$$

we have

$$\sigma_{xP} = \sigma_{x\pi^+} \,, \qquad \vec{v}_P = 0 \,, \qquad \lambda_P = \Sigma_P = 0 \,, \tag{2.82}$$

which imply

$$\xi = 0 \,. \tag{2.83}$$

In this limit the different massive neutrino components have the same energy.

Since $\sigma_{pP} = \sigma_{p\pi^+} \neq 0$, the production process has a momentum uncertainty, but eq. (2.19) implies that both the pion and the muon have no energy uncertainty. From Eqs. (2.67) and (2.70) it follows that each massive neutrino component in the state (2.32) has definite energy and momentum along the direction of propagation, without any spread.

This is due to the fact that the relations (2.82) imply that $S_a^P(\vec{p})$ in eq. (2.34) is infinite, suppressing the production of ν_a, unless $E_{\nu_a}(\vec{p}) = E_P$ exactly. This constraint can be satisfied simultaneously for different values of the index a taking different values of $\vec{p}$. The momentum uncertainty of the pion is necessary in order to allow the coherent production of a state with different massive neutrino components with different values of $\vec{p}$. For each massive neutrino component there is only one energy $E_a = E_P$, equal for all components, and the corresponding momentum $p_a = \sqrt{E_P^2 - m_a^2}$. This implies that along the direction of propagation each massive neutrino component in the state (2.32) is a plane wave and not a wave packet.

Equations (2.68) and (2.69) imply that the uncertainties of the components of the momentum orthogonal to the direction of propagation are equal to σ_{pP}. These uncertainties are allowed because they generate an energy uncertainty of higher order in σ_{pP}/E_P, that has been neglected in our formalism. Actually, as we have already remarked in the discussion of the previous example, these uncertainties are necessary in order to localize the massive neutrino components along the direction of propagation.

The case under consideration has some similarity with the one considered in ref. [27] by Grimus and Stockinger, in which it was assumed that the particles that take part in the production process (as well as those participating to the detection process) are in bound states with definite energy or are described by plane waves with definite energy. Although the physical picture in ref. [27] is different from the example under consideration, in which the localized pion at rest is free, in both cases there is no energy uncertainty in the production process and different massive neutrino components have the same energy because of exact energy conservation. In this sense, the case considered in ref. [27] can be considered effectively as a limiting case of the general wave-packet treatment considered here, as done in ref. [26] comparing the model of ref. [27] with the wave-packet model with virtual intermediate neutrinos discussed in refs. [17, 18].

We think that in some cases it may be important to consider the fact that some particles are in bound states, but an appropriate description of such a case should take into account also the fact that the bound states (as atomic nuclei) are localized, leading to a wave packet description. We also think that the description of some particles taking part to the production (or detection) process by plane waves is unrealistic, because neutrinos are usually produced in dense media, where these particles are localized by interactions.

In conclusion of this subsection we would like to remark that, although the equal energy limit that we have considered is realizable in principle, we think that in practice it is rather unlikely since the pion (or in general the initial particle P_I) must decay exactly at rest and the produced muon (or in general the final particles P_F and ℓ_α^+) must be completely unlocalized. If the pion does not decay at rest in the reference frame of the observer, it is possible to boost the reference frame to the one in which the pion is at rest, but the pion velocity in the reference frame of the observer must be known with high accuracy. This information is usually not available, for example in the existing accelerator and atmospheric neutrino experiments in which neutrinos are produced by pion decay in flight.

2.4 Realistic case

In a realistic experimental setup the localizations of the pion and muon are of the same order of magnitude. Indeed, the typical neutrino production in pion decay at rest[7] occurs in a medium, where both the pion and muon are localized by interactions with the surrounding atoms. Let us consider, for example,

$$\sigma_{x\pi^+} \simeq \sigma_{x\mu^+} \simeq 2\sigma_{xP}\,, \tag{2.84}$$

which leads to

$$\vec{v}_P \simeq \frac{\vec{v}_{\mu^+}}{2} = -\frac{v_{\mu^+}}{2}\vec{\ell}\,, \qquad \Sigma_P \simeq \frac{v_{\mu^+}^2}{2}\,, \qquad \lambda_P \simeq \frac{v_{\mu^+}^2}{4}\,. \tag{2.85}$$

In this case we have

$$\xi \simeq \frac{\frac{1}{2}\left(1-\frac{m_{\pi^+}^2}{m_{\mu^+}^2}\right)}{1+\frac{1}{4}\left(1+\frac{m_{\pi^+}^2}{m_{\mu^+}^2}\right)^2} \simeq 0.13\,. \tag{2.86}$$

Hence we see that the value of ξ in a realistic situation is of the same order of magnitude as that in eq. (2.79), corresponding to exact energy-momentum conservation in the production process in the case of pion decay at rest and different muon momenta for each massive neutrino.

The condition (2.64) in this case becomes

$$m_a^2 \lesssim \sigma_{pP} E \sqrt{1 + v_{\mu^+} + \frac{v_{\mu^+}^2}{2}} \sim \sigma_{p\mu^+} E\,. \tag{2.87}$$

If the pion decay occurs in normal matter, σ_{xP} is given approximately by the inter-atomic distance, $\sigma_{xP} \sim 10^{-8}\,\mathrm{cm}$, corresponding to $\sigma_{pP} \sim 10^3\,\mathrm{eV}$, the condition (2.87) numerically reads $m_a^2 \lesssim 10^7\,\mathrm{eV}^2$, that is certainly satisfied.

Since in this realistic case $\vec{v}_P$ and λ_P do not have extreme values, from eqs. (2.67)–(2.70) one can see that the energy and momentum uncertainties of the neutrino state are of the order of σ_{pP}, about $10^3\,\mathrm{eV}$ for pion decay in normal matter.

3. Detection

Let us consider neutrino detection through the charged-current weak process

$$\nu_\beta + D_I \to D_F + \ell_\beta^-\,, \tag{3.1}$$

at a space-time distance $(\vec{L}, T)$ from the production process.

[7] In accelerator experiments in which neutrinos are produced by pion decay in flight the localization of the pion and muon are given by the dimensions of the decay tunnel. In solar neutrino experiments electron neutrinos are produced in the core of the sun where all the particles taking part to the production process are localized by interactions with the dense medium. In atmospheric neutrino experiments neutrinos are produced by pion and muon decay in flight in the atmosphere, where pions, muons and electrons are localized by the interactions with air. In reactor neutrinos experiments electron antineutrinos are produced by the decays of heavy elements in the dense reactor core where the heavy nuclei and electrons are localized by interactions with the medium.

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The state $|\nu_\alpha\rangle$ in eq. (2.32) describes the neutrino produced in the process (2.3) at the origin of the space-time coordinates. Since we want to describe with the same formalism the detection of the neutrino through the process (3.1) occurring at a space-time distance $(\vec{L}, T)$ from the production process, we must translate the origin to the detection space-time point. Hence, the neutrino state relevant for the detection process is obtained by acting on $|\nu_\alpha\rangle$ with the space-time translation operator $\exp(-i\widehat{E}T + i\widehat{\vec{P}} \cdot \vec{L})$, where $\widehat{E}$ and $\widehat{\vec{P}}$ are the energy and momentum operators, respectively. The resulting state is

$$|\nu_\alpha(\vec{L}, T)\rangle = N_\alpha \sum_a U_{\alpha a}^* \int \mathrm{d}^3p \exp\left(-iE_{\nu_a}(\vec{p})T + i\vec{p} \cdot \vec{L}\right) e^{-S_a^P(\vec{p})} \sum_h \mathcal{A}_a^P(\vec{p}, h) |\nu_a(\vec{p}, h)\rangle \, . \tag{3.2}$$

The amplitude of interaction of the neutrino state $|\nu_\alpha(\vec{L}, T)\rangle$ in the detection process (3.1) is

$$\mathcal{A}_{\alpha\beta}(\vec{L}, T) = \langle D_F, \ell_\beta^- | -i \int \mathrm{d}^4x \mathcal{H}_I^D(x) | D_I, \nu_\alpha(\vec{L}, T)\rangle \, , \tag{3.3}$$

where

$$\begin{aligned} \mathcal{H}_I^D(x) &= \frac{G_F}{\sqrt{2}} \bar{\ell}_\beta(x) \gamma^\rho (1 - \gamma_5) \nu_\beta(x) J_\rho^D(x) \\ &= \frac{G_F}{\sqrt{2}} \sum_b U_{\beta b} \bar{\ell}_\beta(x) \gamma^\rho (1 - \gamma_5) \nu_b(x) J_\rho^D(x) \, , \end{aligned} \tag{3.4}$$

is the effective interaction hamiltonian that describes the detection process, and $J_\rho^D(x)$ is the weak charged current that describes the transition $D_I \to D_F$.

For simplicity, we assume that the particles that take part to the detection process are described by the gaussian wave-packet states (2.8) with $\chi = D_I, D_F, \ell_\beta^-$. Using the neutrino state (3.2) and the Fourier expansion (2.20) for the lepton fields, respectively, we obtain

$$\begin{aligned} \mathcal{A}_{\alpha\beta}(\vec{L}, T) \propto & \sum_a U_{\alpha a}^* U_{\beta a} \int \mathrm{d}^4x \sum_h \int \mathrm{d}^3p \mathcal{A}_a^P(\vec{p}, h) e^{-S_a^P(\vec{p})} \exp\left[-iE_{\nu_a}(\vec{p})T + i\vec{p} \cdot \vec{L}\right] \times \\ & \times \int \mathrm{d}^3p'_{D_F} \psi_{D_F}^*(\vec{p}'_{D_F}; \vec{p}_{D_F}, \sigma_{pD_F}) \int \mathrm{d}^3p'_{\ell_\beta^-} \psi_{\ell_\beta^-}^*(\vec{p}'_{\ell_\beta^-}; \vec{p}_{\ell_\beta^-}, \sigma_{\ell_\beta^-}) \times \\ & \times \int \mathrm{d}^3p'_{D_I} \psi_{D_I}(\vec{p}'_{D_I}; \vec{p}_{D_I}, \sigma_{pD_I}) \bar{u}_{\ell_\beta^-}(\vec{p}_{\ell_\beta^-}, h_{\ell_\beta^-}) \gamma^\rho (1 - \gamma_5) u_{\nu_a}(\vec{p}, h) \times \\ & \times J_\rho^D(\vec{p}'_{D_F}, h_{D_F}; \vec{p}'_{D_I}, h_{D_I}) e^{i(p'_{D_F} + p'_{\ell_\beta^-} - p'_{D_I} - \vec{p})x} \, , \end{aligned} \tag{3.5}$$

where $p^0 = E_{\nu_a}(\vec{p})$, $p'^0_{D_I} = E_{D_I}(\vec{p}'_{D_I})$, $p'^0_{D_F} = E_{D_F}(\vec{p}'_{D_F})$, $p'^0_{\ell_\beta^-} = E_{\ell_\beta^-}(\vec{p}'_{\ell_\beta^-})$, and $J_\rho^D(\vec{p}'_{D_F}, h_{D_F}; \vec{p}'_{D_I}, h_{D_I}) = \langle D_F(\vec{p}'_{D_F}, h_{D_F}) | J_\rho^D(0) | D_I(\vec{p}'_{D_I}, h_{D_I})\rangle$. Following the same method as that used for the production process (see eq. (2.24)), we perform the integrals over $\mathrm{d}^3p'_{D_F}$, $\mathrm{d}^3p'_{\ell_\beta^-}$ and

$\mathrm{d}^3p'_{D_I}$ with a saddle-point approximation through the approximation (2.12), obtaining

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_a U^*_{\alpha a} U_{\beta a} \sum_h \int \mathrm{d}^3p \mathcal{A}^P_a(\vec{p},h)\mathcal{A}^D_a(\vec{p},h) e^{-S^P_a(\vec{p})} \exp\left[-iE_{\nu_a}(\vec{p})T + i\vec{p}\cdot\vec{L}\right] \times$$
$$\times \int \mathrm{d}^4x \exp\left[-i(E_{\nu_a}(\vec{p}) - E_D)t + i(\vec{p}-\vec{p}_D)\vec{x} - \frac{\vec{x}^2 - 2\vec{v}_D\cdot\vec{x}t + \Sigma_D t^2}{4\sigma^2_{xD}}\right], \quad (3.6)$$

with

$$E_D \equiv E_{D_F} + E_{\ell^-_\beta} - E_{D_I}\,, \quad (3.7)$$
$$\vec{p}_D \equiv \vec{p}_{D_F} + \vec{p}_{\ell^-_\beta} - \vec{p}_{D_I}\,, \quad (3.8)$$
$$\frac{1}{\sigma^2_{xD}} \equiv \frac{1}{\sigma^2_{xD_I}} + \frac{1}{\sigma^2_{xD_F}} + \frac{1}{\sigma^2_{x\ell^-_\beta}}\,, \quad (3.9)$$
$$\vec{v}_D \equiv \sigma^2_{xD}\left(\frac{\vec{v}_{D_I}}{\sigma^2_{xD_I}} + \frac{\vec{v}_{D_F}}{\sigma^2_{xD_F}} + \frac{\vec{v}_{\ell^-_\beta}}{\sigma^2_{x\ell^-_\beta}}\right)\,, \quad (3.10)$$
$$\Sigma_D \equiv \sigma^2_{xD}\left(\frac{\vec{v}^2_{D_I}}{\sigma^2_{xD_I}} + \frac{\vec{v}^2_{D_F}}{\sigma^2_{xD_F}} + \frac{\vec{v}^2_{\ell^-_\beta}}{\sigma^2_{x\ell^-_\beta}}\right)\,, \quad (3.11)$$
$$\mathcal{A}^D_a(\vec{p},h) \equiv \bar{u}_{\ell^-_\beta}(\vec{p}_{\ell^-_\beta}, h_{\ell^-_\beta})\gamma^\rho\left(1-\gamma_5\right)u_{\nu_a}(\vec{p},h) J^D_\rho(\vec{p}'_{D_F}, h_{D_F}; \vec{p}'_{D_I}, h_{D_I})\,. \quad (3.12)$$

As expected, the overall spatial width σ_{xD} of the production process is dominated by the smallest among the spatial widths of the participating particles. As the corresponding quantities in the production process, $|\vec{v}_D|$ and Σ_D are limited by

$$0 \le |\vec{v}_D| \le 1\,, \qquad 0 \le \Sigma_D \le 1\,. \quad (3.13)$$

The gaussian integral over d^4x leads to

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_a U^*_{\alpha a} U_{\beta a} \sum_h \int \mathrm{d}^3p \mathcal{A}^P_a(\vec{p},h)\mathcal{A}^D_a(\vec{p},h) e^{-S_a(\vec{p})} \exp\left[-iE_{\nu_a}(\vec{p})T + i\vec{p}\cdot\vec{L}\right]\,, \quad (3.14)$$

where

$$S_a(\vec{p}) = S^P_a(\vec{p}) + S^D_a(\vec{p})\,, \quad (3.15)$$

and $S^D_a(\vec{p})$ has the same structure as $S^P_a(\vec{p})$ given in eq. (2.34), with the quantities relative to the production process replaced by the corresponding ones relative to the detection process:

$$S^D_a(\vec{p}) \equiv \frac{(\vec{p}_D - \vec{p})^2}{4\sigma^2_{pD}} + \frac{[(E_D - E_{\nu_a}(\vec{p})) - (\vec{p}_D - \vec{p})\cdot\vec{v}_D]^2}{4\sigma^2_{pD}\lambda_D}\,, \quad (3.16)$$

with

$$\lambda_D \equiv \Sigma_D - \vec{v}^2_D\,, \quad (3.17)$$

limited by

$$0 \le \lambda_D \le 1\,, \quad (3.18)$$

and the momentum uncertainty σ_{pD} defined by

$$\sigma_{xD}\,\sigma_{pD} = \frac{1}{2}\,, \tag{3.19}$$

which is dominated by the largest of the momentum uncertainties of D_I, D_F and ℓ_β^-:

$$\sigma_{pD}^2 = \sigma_{pD_I}^2 + \sigma_{pD_F}^2 + \sigma_{p\ell_\beta^-}^2\,. \tag{3.20}$$

In general the dominant momentum contribution to the integration over d^3p (3.14) does not come from the stationary point of $S_a^P(\vec{p})$, but from the stationary point of $S_a(\vec{p})$, which takes into account also the momentum and energy uncertainties of the detection process. If these uncertainties are smaller than those of the production process, the detection process picks up as dominant contribution to the flavor-changing amplitude (3.14) a value of the neutrino momentum in the wave packet (3.2) that is significantly different from $\vec{p}_a$, which corresponds to the stationary point of $S_a^P(\vec{p})$.

We denote by $\vec{k}_a$ the momentum corresponding to the stationary point of $S_a(\vec{p})$,

$$\left.\frac{\partial S_a}{\partial \vec{p}}\right|_{\vec{p}=\vec{k}_a} = 0\,, \tag{3.21}$$

which gives the dominant contribution to the transition amplitude (3.14). The value of $\vec{k}_a$ is given by

$$\begin{aligned}
&\frac{\vec{p}_P - \vec{k}_a}{\sigma_{pP}^2} + \frac{E_P - \varepsilon_a - \left(\vec{p}_P - \vec{k}_a\right)\cdot\vec{v}_P}{\sigma_{pP}^2\,\lambda_P}\,(\vec{u}_a - \vec{v}_P) + \\
&+\frac{\vec{p}_D - \vec{k}_a}{\sigma_{pD}^2} + \frac{E_D - \varepsilon_a - \left(\vec{p}_D - \vec{k}_a\right)\cdot\vec{v}_D}{\sigma_{pD}^2\,\lambda_D}\,(\vec{u}_a - \vec{v}_D) = 0\,,
\end{aligned} \tag{3.22}$$

with

$$\varepsilon_a \equiv E_{\nu_a}(\vec{k}_a) = \sqrt{\vec{k}_a^{\,2} + m_a^2}\,, \tag{3.23}$$

$$\vec{u}_a \equiv \left.\frac{\partial E_{\nu_a}(\vec{p})}{\partial \vec{p}}\right|_{\vec{p}=\vec{k}_a} = \frac{\vec{k}_a}{\varepsilon_a}\,. \tag{3.24}$$

We have chosen the notation $\vec{k}_a$, ε_a and $\vec{u}_a$ in order to emphasize that in general these quantities are not the average momenta, energies and velocities of the neutrino wave packets propagating between production and detection, denoted by $\vec{p}_a$, E_a and $\vec{v}_a$, which are determined only by the production process, as follows from Eqs. (2.45)–(2.47). The quantities $\vec{k}_a$, ε_a and $\vec{u}_a$ are approximately equal to the average neutrino wave packets momenta, energies and velocities $\vec{p}_a$, E_a and $\vec{v}_a$ only if $\sigma_{pD}^2 \gg \sigma_{pP}^2$ and $\sigma_{pD}^2\,\lambda_D \gg \sigma_{pP}^2\,\lambda_P$. In other words, the properties of the neutrino wave packets can be measured only with a detection process having a relatively large momentum and energy uncertainty, which correspond to a relatively sharp spatial and temporal localization.

Before solving eq. (3.22), we notice that the amplitude (3.14) is not suppressed only if

$$S_a(\vec{k}_a) \lesssim 1 \tag{3.25}$$

for all values of a. The inequality (3.25), together with eq. (3.22), constraint the possible values of $\vec{k}_a$, $\vec{p}_P$, E_P, $\vec{p}_D$ and E_D. Since both $S_a^P(\vec{k}_a)$ and $S_a^D(\vec{k}_a)$ are positive, we have the conditions $S_a^P(\vec{k}_a) \ll 1$ and $S_a^D(\vec{k}_a) \ll 1$ for all values of a. The first inequality is similar to the condition (2.54), but now it concerns the momenta $\vec{k}_a$ and energies ε_a that are relevant for the flavor transition amplitude (3.14).

Similarly to what we have done in the discussion of the production process (see eq. (2.60)), we consider

$$\vec{p}_P = E_P \vec{\ell}\,, \qquad \vec{p}_D = E_D \vec{\ell}\,, \qquad E_P = E_D\,, \tag{3.26}$$

which corresponds to exact energy and momentum conservation in the production and detection processes in the case of massless neutrinos. Here $\vec{\ell} \equiv \vec{L}/|\vec{L}|$ is the unit vector in the direction of propagation of the neutrino from the production to the detection processes. It is possible to consider deviations from the relations (3.26) compatible with the inequality (3.25), but such deviations would introduce many complications in the following formalism, without further insights in the physics of neutrino oscillations.

Equation (3.22) implies that the massive neutrino momenta $\vec{k}_a$ must be aligned in the $\vec{\ell}$ direction:

$$\vec{k}_a = k_a \vec{\ell}\,. \tag{3.27}$$

We solve eq. (3.22) in the relativistic approximation

$$\varepsilon_a \simeq E + \rho \frac{m_a^2}{2E}\,, \tag{3.28}$$

$$k_a \simeq E - (1-\rho)\,\frac{m_a^2}{2E}\,. \tag{3.29}$$

At zeroth order in m_a^2/E^2 we obtain

$$E = E_P = E_D\,, \tag{3.30}$$

and at first order

$$\rho = \frac{\frac{1}{\sigma_p^2} - \frac{\vec{\ell}\cdot\vec{v}_P\left(1-\vec{\ell}\cdot\vec{v}_P\right)}{\sigma_{pP}^2\lambda_P} - \frac{\vec{\ell}\cdot\vec{v}_D\left(1-\vec{\ell}\cdot\vec{v}_D\right)}{\sigma_{pD}^2\lambda_D}}{\frac{1}{\sigma_p^2} + \frac{\left(1-\vec{\ell}\cdot\vec{v}_P\right)^2}{\sigma_{pP}^2\lambda_P} + \frac{\left(\vec{\ell}\cdot\vec{v}_D-1\right)^2}{\sigma_{pD}^2\lambda_D}}\,, \tag{3.31}$$

with

$$\frac{1}{\sigma_p^2} = \frac{1}{\sigma_{pP}^2} + \frac{1}{\sigma_{pD}^2}\,. \tag{3.32}$$

Notice that in the production and detection processes the squared momentum uncertainties add, as shown in Eqs. (2.27) and (3.9), whereas in the total amplitude the inverses of the squared momentum uncertainties of the production and detection processes add. This is expected on the basis of simple physical arguments. Indeed, a large momentum uncertainty of a particle must increase the total momentum uncertainty in the corresponding process, whereas a small momentum uncertainty in one of the two processes constraints the momentum uncertainty of the neutrino connecting the two processes leading to a restriction of the momentum uncertainty in the other process. On the other hand, in the production and detection processes the inverses of the squared space uncertainties add (see eqs. (2.27) and (3.9)), whereas in the total amplitude the squared space uncertainties of the production and detection processes add:

$$\sigma_x^2 = \sigma_{xP}^2 + \sigma_{xD}^2 \,, \tag{3.33}$$

with σ_x defined by the relation

$$\sigma_x \sigma_p = \frac{1}{2} \,. \tag{3.34}$$

Also the behavior in eq. (3.33) is expected on the basis of simple physical arguments: a small space uncertainty of a particle localizes better the corresponding process, whereas a large space uncertainty of one of the two processes increases the coherence of the overall process.

In the relativistic approximation $S_a(\vec{k}_a)$ is given by

$$S_a(\vec{k}_a) = \zeta \left(\frac{m_a^2}{4E\sigma_p} \right)^2 , \tag{3.35}$$

with

$$\zeta = \frac{\frac{1}{\sigma_p^2}\left(\frac{1}{\sigma_{pP}^2\lambda_P}+\frac{1}{\sigma_{pD}^2\lambda_D}\right)+\left(\frac{1-\vec{\ell}\cdot\vec{v}_P}{\sigma_{pP}^2\lambda_P}+\frac{1-\vec{\ell}\cdot\vec{v}_D}{\sigma_{pD}^2\lambda_D}\right)^2+\frac{\left[\vec{\ell}\cdot(\vec{v}_P-\vec{v}_D)\right]^2}{\sigma_{pP}^2\lambda_P\sigma_{pD}^2\lambda_D}\left(2+\frac{\sigma_p^2}{\sigma_{pP}^2}\frac{\left(1-\vec{\ell}\cdot\vec{v}_P\right)^2}{\lambda_P}+\frac{\sigma_p^2}{\sigma_{pD}^2}\frac{\left(1-\vec{\ell}\cdot\vec{v}_D\right)^2}{\lambda_D}\right)}{\left[\frac{1}{\sigma_p^2}+\frac{\left(1-\vec{\ell}\cdot\vec{v}_P\right)^2}{\sigma_{pP}^2\lambda_P}+\frac{\left(1-\vec{\ell}\cdot\vec{v}_D\right)^2}{\sigma_{pD}^2\lambda_D}\right]^2} . \tag{3.36}$$

It follows that in the relativistic approximation the condition (3.25) reads

$$m_a^2 \lesssim \sigma_p E \zeta^{-1/2} \,. \tag{3.37}$$

If this inequality is satisfied for all values of the index a, the choices (3.26) are acceptable and the different extremely relativistic massive neutrino components contribute coherently to the flavor-changing transition amplitude.

Let us return to the calculation of the transition amplitude. The integral over d^3p in eq. (3.14) can be performed with a saddle-point approximation around the stationary point $\vec{k}_a$ of $S_a(\vec{p})$, leading to

$$\begin{aligned} \mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_a U_{\alpha a}^* U_{\beta a} \sum_h \frac{\mathcal{A}_a^P(\vec{k}_a,h)\mathcal{A}_a^D(\vec{k}_a,h)}{\sqrt{\mathrm{Det}\Omega_a}} e^{-S_a(\vec{k}_a)} \\ \times \exp\left[-i\varepsilon_a T + i\vec{k}_a\cdot\vec{L} - \frac{1}{2}\left(L^j - u_a^j T\right)\left(\Omega_a^{-1}\right)^{jk}\left(L^k - u_a^k T\right)\right] , \end{aligned} \tag{3.38}$$

where

$$\begin{aligned}
\Omega_a^{jk} &\equiv \left. \frac{\partial^2 S_a(\vec{p})}{\partial p^j \partial p^k} \right|_{\vec{p}=\vec{k}_a} \\
&= \frac{\delta^{jk}}{2\sigma_p^2} + \frac{\left(v_P^j - u_a^j\right)\left(v_P^k - u_a^k\right)}{2\sigma_{pP}^2 \lambda_P} - \frac{\left[(E_P - \varepsilon_a) - \left(\vec{p}_P - \vec{k}_a\right)\cdot \vec{v}_P\right]}{2\sigma_{pP}^2 \lambda_P} \frac{\delta^{jk} - u_a^j u_a^k}{\varepsilon_a} \\
&\quad + \frac{\left(v_D^j - u_a^j\right)\left(v_D^k - u_a^k\right)}{2\sigma_{pD}^2 \lambda_D} - \frac{\left[(E_D - \varepsilon_a) - \left(\vec{p}_D - \vec{k}_a\right)\cdot \vec{v}_D\right]}{2\sigma_{pD}^2 \lambda_D} \frac{\delta^{jk} - u_a^j u_a^k}{\varepsilon_a} .
\end{aligned} \tag{3.39}$$

The factors

$$-\frac{1}{2}\left(L^j - u_a^j T\right)\left(\Omega_a^{-1}\right)^{jk}\left(L^k - u_a^k T\right) \tag{3.40}$$

in the exponential of eq. (3.38) do not suppress the transition amplitude only if $\vec{L}$ is aligned with the common direction $\vec{\ell}$ of the momenta $\vec{k}_a$, apart from a deviation of the order of Ω_a/L, which is very small if the localizations of the production and detection processes are much smaller than the source-detector distance, a condition which is necessary for the observation of neutrino oscillations and which is satisfied in all neutrino oscillation experiments. Therefore, we consider $\vec{L} = L\vec{\ell}$ and we align $\vec{\ell}$ along the direction of the x axis. Thus, the amplitude (3.38) can be written as

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_a U_{\alpha a}^* U_{\beta a} \sum_h \frac{\mathcal{A}_a^P(\vec{k}_a,h)\mathcal{A}_a^D(\vec{k}_a,h)}{\sqrt{\mathrm{Det}\Omega_a}} e^{-S_a(\vec{k}_a)} \exp\left[-i\varepsilon_a T + i k_a L - \frac{(L-u_a T)^2}{4\eta_a^2}\right], \tag{3.41}$$

where

$$\eta_a \equiv \sqrt{\frac{1}{2(\Omega_a^{-1})^{xx}}} \tag{3.42}$$

are the spatial coherence widths.

In the relativistic approximation the product $\mathcal{A}_a^P(\vec{k}_a,+)\mathcal{A}_a^D(\vec{k}_a,+)$ corresponding to the positive helicity component of the massive neutrino ν_a is suppressed by the ratio m_a^2/E^2 with respect to the product $\mathcal{A}_a^P(\vec{k}_a,-)\mathcal{A}_a^D(\vec{k}_a,-)$ corresponding to the negative helicity component and can be neglected. In the same approximation, the factors

$$\frac{\mathcal{A}_a^P(\vec{k}_a,-)\mathcal{A}_a^D(\vec{k}_a,-)}{\sqrt{\mathrm{Det}\Omega_a}} e^{-S_a(\vec{k}_a)} , \tag{3.43}$$

can be approximated with their value in the case of massless neutrinos, can be extracted out of the sum over the index a and absorbed in the overall normalization factor of the flavor transition amplitude. The factorization of these quantities allows the calculation of the flavor transition amplitude independently from the production and detection rates. The oscillation probability obtained from this flavor transition amplitude can be used in the usual calculations of event rates in neutrino oscillation experiments, given by the product of the neutrino flux calculated for massless neutrinos, times the oscillation probability, times the detection cross section calculated for massless neutrinos (see [8]–[11]).

It is important to notice, however, that a special care is needed for the factors $e^{-S_a(\vec{k}_a)}$, to make sure that they do not suppress the contribution of the corresponding massive neutrino component. The factors $e^{-S_a(\vec{k}_a)}$ can be approximated with their value in the case of massless neutrinos only if

$$S_a(\vec{k}_a) \ll 1 \tag{3.44}$$

for all values of the index a. Hence, the condition (3.37) must be strengthened to

$$m_a^2 \ll \sigma_p E \zeta^{-1/2} \tag{3.45}$$

for all values of a. Let us emphasize that this condition is necessary for the unsuppressed production and detection of the different extremely relativistic massive neutrino components whose interference generates neutrino oscillations. In principle, it is possible to consider degenerate neutrinos for which the massless approximation is not appropriate, but as noted in footnote 5, this case is likely to be irrelevant in practice. In any case, considering a case in which the massless approximation is not appropriate would require the inclusion of the neutrino mass effect, which are normally neglected, in the calculation of the production and detection rates.

If the condition (3.45) is satisfied for all values of a, the transition amplitude in the relativistic approximation can be written as

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \sum_a U_{\alpha a}^* U_{\beta a} \exp\left[-i\varepsilon_a T + i k_a L - \frac{(L-u_a T)^2}{4\eta^2} \right] . \tag{3.46}$$

Here ε_a and k_a are given by their relativistic approximations (3.28) and (3.29), and

$$u_a \simeq 1 - \frac{m_a^2}{2E^2} . \tag{3.47}$$

The coherence widths η_a have been approximated with their value η at zeroth order in m_a^2/E^2.

The value of η is given by the inversion of the zeroth order approximation in m_a^2/E^2 of the symmetric matrix Ω_a in eq. (3.39),

$$\Omega_a^{jk} = \frac{\delta^{jk}}{2\sigma_p^2} + \frac{\left(v_P^j - \delta^{jx}\right)\left(v_P^k - \delta^{kx}\right)}{2\sigma_{pP}^2\lambda_P} + \frac{\left(v_D^j - \delta^{jx}\right)\left(v_D^k - \delta^{kx}\right)}{2\sigma_{pD}^2\lambda_D} + \mathrm{O}\left(\frac{m_a^2}{E^2}\right) . \tag{3.48}$$

We obtained

$$\eta^2 = \omega \sigma_x^2 , \tag{3.49}$$

with

$$\begin{aligned} \omega = & \left\{ 1 + \sigma_p^2 \left[\frac{(v_P^x - 1)^2 + (v_P^y)^2 + (v_P^z)^2}{\sigma_{pP}^2 \lambda_P} + \frac{(v_D^x - 1)^2 + (v_D^y)^2 + (v_D^z)^2}{\sigma_{pD}^2 \lambda_D} \right] \right. \\ & \left. + \sigma_p^4 \frac{\left[(v_P^x - 1)\, v_D^y - (v_D^x - 1)\, v_P^y\right]^2 + \left[(v_P^x - 1)\, v_D^z - (v_D^x - 1)\, v_P^z\right]^2 + \left(v_P^y v_D^z - v_P^z v_D^y\right)^2}{\sigma_{pP}^2 \lambda_P \sigma_{pD}^2 \lambda_D} \right\} \times \\ & \times \left\{ 1 + \sigma_p^2 \left[\frac{(v_P^y)^2 + (v_P^z)^2}{\sigma_{pP}^2 \lambda_P} + \frac{(v_D^y)^2 + (v_D^z)^2}{\sigma_{pD}^2 \lambda_D} \right] + \sigma_p^4 \frac{\left(v_P^y v_D^z - v_P^z v_D^y\right)^2}{\sigma_{pP}^2 \lambda_P \sigma_{pD}^2 \lambda_D} \right\}^{-1} . \end{aligned} \tag{3.50}$$

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Let us notice that the width η obtained in eq. (3.49) is somewhat different from the corresponding width obtained in ref. [18]. The difference stems from the fact that Ω_a in eq. (3.39) is a matrix, because of the integration over d^3p in eq. (3.14), whereas the width in eq. (18) of ref. [18] is a number obtained from the integration over $\mathrm{d}q^0$ in eq. (15) of ref. [18].

In the limit $\sigma_{pP} \ll \sigma_{pD}$ the total momentum uncertainty σ_p and the coherence width η are dominated by the production process, and $\rho \simeq \xi$, with ξ given in eq. (2.62). This happens if the production process is much less localized than the detection process. In this case $\vec{k}_a \simeq \vec{p}_a$, the average momentum of the massive neutrino component ν_a of the state $|\nu_\alpha\rangle$ created in the production process. In other words, if the detection process is much more localized than the production process, i.e. if the momentum uncertainty of the detection process is much larger than that of the production process, the transition amplitude (3.46) depends only on the properties of the neutrino created in the production process.

However, in general, σ_p, η and ρ depend on both the production and detection processes and in the limit $\sigma_{pD} \ll \sigma_{pP}$, in which the detection process is much less localized than the production process, they are dominated by the detection process.

The present derivation of the transition amplitude in neutrino oscillation experiments is more complete than the simple quantum-mechanical model presented in refs. [15, 16], in which the energies and momenta of the massive neutrino components contributing to the transition amplitude are undetermined and have to be assumed.

Formally, the transition amplitude (3.14) can be derived by projecting the state $|\nu_\alpha(\vec{L},T)\rangle$ in eq. (3.2) on the state

$$|\nu_\beta\rangle = N_\beta \sum_a U_{\beta a}^* \int \mathrm{d}^3p e^{-S_a^D(\vec{p})} \sum_h \mathcal{A}_a^D(\vec{p},h) |\nu_a(\vec{p},h)\rangle \,, \tag{3.51}$$

representing the detection process:

$$\mathcal{A}_{\alpha\beta}(\vec{L},T) \propto \langle \nu_\beta | \nu_\alpha(\vec{L},T) \rangle \,, \tag{3.52}$$

as in the quantum-mechanical derivation of the flavor transition probability in neutrino oscillation experiments. However, the calculation of the coefficients of the massive neutrino components in the states $|\nu_\alpha(\vec{L},T)\rangle$ and $|\nu_\beta\rangle$ require the quantum field theoretical framework that we have adopted.

4. Transition probability

In this section we calculate and discuss the transition probability in space obtained from the average of

$$P_{\alpha\beta}(\vec{L},T) \propto |\mathcal{A}_{\alpha\beta}(\vec{L},T)|^2 \tag{4.1}$$

over the unmeasured propagation time T, as done in refs. [15]–[18]. Let us notice, however, that although in present neutrino oscillation experiments only the source-detector distance L is known, in the future it may be possible to measure also the propagation time T, leading to the experimental relevance of the space-time dependent transition probability (4.1) (see refs. [43, 44]).

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From the gaussian integration over dT of $P_{\alpha\beta}(\vec{L},T)$, in the relativistic approximation we obtain[8]

$$P_{\alpha\beta}(\vec{L}) = \sum_a |U_{\alpha a}|^2 |U_{\beta a}|^2 + \\ +2\mathrm{Re}\sum_{a>b} U_{\alpha a}^* U_{\beta a} U_{\alpha b} U_{\beta b}^* \exp\left[-2\pi i \frac{L}{L_{ab}^{\mathrm{osc}}} - \left(\frac{L}{L_{ab}^{\mathrm{coh}}}\right)^2 - 2\pi^2 \rho^2 \omega \left(\frac{\sigma_x}{L_{ab}^{\mathrm{osc}}}\right)^2\right], \tag{4.2}$$

where

$$L_{ab}^{\mathrm{osc}} = \frac{4\pi E}{|\Delta m_{ab}^2|} \tag{4.3}$$

are the oscillation lengths, and

$$L_{ab}^{\mathrm{coh}} = \frac{4\sqrt{2\omega}E^2}{|\Delta m_{ab}^2|}\sigma_x \tag{4.4}$$

are the coherence lengths.

The transition probability (4.2) has the same form as that obtained in ref. [18] in quantum field theory with virtual propagating neutrinos. The value of ρ in quantum field theory with virtual propagating neutrinos, derived in ref. [26], is the same as the one derived here (see eq. (3.31)). Instead, the value of ω is somewhat different from that derived in ref. [18], as already remarked after eq. (3.50).

The last term in the exponential of the transition probability (4.2) suppresses the corresponding oscillatory term unless the localization of the production and detection processes is much smaller than the oscillation length,

$$\sigma_x \ll L_{ab}^{\mathrm{osc}} . \tag{4.5}$$

This is due to the average of the transition probability (4.1) over the unmeasured propagation time T. Indeed, a spatial uncertainty σ_x imply a similar time uncertainty, as one can understand from the relation between the energy and momentum uncertainties in eq. (2.19), or by noticing that the spatial region in which a process can occur coherently must be causally connected. If the time uncertainty, of the order of σ_x, is larger than the oscillation length, the average of an oscillatory term over the propagation time T is an average over all oscillation phases which depend on T, and the result is zero.

However, as discussed in ref. [26], the condition (4.5) for the observability of neutrino oscillations is not necessary if

$$\rho^2\omega = 0 . \tag{4.6}$$

As shown in the example presented in subsection 4.2, this can be obtained with $\rho = 0$. In this case, since the different massive neutrino components have the same energy $\varepsilon_a = E$, the oscillation phases do not depend on T and the average over T of the space-time dependent transition probability (4.1) has no effect. Furthermore, in this case the coherence lengths L_{ab}^{coh} in eq. (4.4) can be increased without limit [26]. However, one must notice that

[8]We order the massive neutrinos by increasing mass: $m_1 \leq m_2 \leq \ldots$. In this way the Δm_{ab}^2's in the sum over $a > b$ are all positive.

this unlimited increase must be obtained by increasing ω and not σ_x, because an unlimited increase of σ_x would bring $\sigma_p \to 0$ and a violation of the condition (3.45) necessary for the coherent production of the different massive neutrino components whose interference generates the oscillations.

In the following Subsections we consider three cases analogous to those considered in Subsections 2.1, 2.3 and 2.4, that illustrate the effects of the detection process. In all these cases we consider the initial detection particle D_I at rest ($\vec{v}_{D_I} = 0$).

4.1 Unlocalized detection process

If the particles taking part to the detection process are unlocalized, we have the limit

$$\sigma_{pD_I} \to 0\,, \qquad \sigma_{pD_F} \to 0\,, \qquad \sigma_{p\ell_\beta^-} \to 0 \quad \Longrightarrow \quad \sigma_{pD} \to 0 \quad \Longrightarrow \quad \sigma_p \to 0\,. \tag{4.7}$$

In this case the condition (3.37) is not satisfied. Moreover, similarly to the case of an unlocalized production process discussed in subsection 2.1, no deviation from eq. (3.26) can satisfy the inequality (3.25) for more than one value of a. Indeed, in the limit (4.7) $S_a^D(\vec{k}_a)$ becomes infinite and suppresses the production of ν_a unless

$$\left(\vec{p}_D - \vec{k}_a\right)^2 + \frac{1}{\lambda_D}\left[E_D - \varepsilon_a - \left(\vec{p}_D - \vec{k}_a\right)\cdot\vec{v}_D\right]^2 = 0\,. \tag{4.8}$$

Since λ_D is positive, eq. (4.8) can be satisfied only if both squares are zero, i.e. if $\vec{p}_D = \vec{k}_a$ and $E_D = \epsilon_a$ exactly. Since this constraint can be satisfied only for one value of the index a, only the corresponding massive neutrino is detected and there are no oscillations, which are due to the interference of different massive neutrino components.

4.2 Equal energy limit

Let us consider a localized initial detection particle D_I at rest, and unlocalized final detection particles,

$$\sigma_{pD_F} = \sigma_{p\ell_\beta^-} = 0\,, \tag{4.9}$$

which imply

$$\sigma_{pD} = \sigma_{pD_I}\,, \qquad \vec{v}_D = 0\,, \qquad \lambda_D = \Sigma_D = 0\,. \tag{4.10}$$

In this case, eq. (2.19) implies that all the detection particles have no energy uncertainty.

Equation (3.31) gives

$$\rho = 0\,. \tag{4.11}$$

Hence, the detection process picks up the momenta $\vec{k}_a$ of the different massive neutrino components of the state $|\nu_\alpha(\vec{L},T)\rangle$ in eq. (3.2) corresponding to the same energy $\varepsilon_a = E$, corresponding to exact energy conservation in the detection process. These momenta are different from the average momenta $\vec{p}_a$ of the massive neutrino wave packets that constitute the state $|\nu_\alpha(\vec{L},T)\rangle$, which are given by eq. (2.45), unless the conditions of subsection 2.3 are satisfied by the production process.

From eq. (3.36) we find

$$\zeta = 1 + \frac{\sigma_p^2}{\sigma_{pP}^2} \frac{\left(\vec{\ell} \cdot \vec{v}_P\right)^2}{\lambda_P} \,. \tag{4.12}$$

Thus, the condition (3.45) can be satisfied by a sufficiently large σ_p.

Since the values in eq. (4.10) imply that

$$\omega \to +\infty \,, \tag{4.13}$$

in the case under consideration

$$L_{ab}^{\mathrm{coh}} \to +\infty \,, \tag{4.14}$$

i.e. the oscillations remain coherent at arbitrarily large distances. In agreement with the arguments presented in ref. [26], the last term in the exponential of eq. (4.2) does not suppress the oscillations because

$$\rho^2 \omega = 0 \,. \tag{4.15}$$

Physically, the infinity of the coherence lengths is due to the fact that the detection process picks up the momentum components with equal energy of the different massive neutrino components. In this case there are no oscillations in time and the average over time of the space-time dependent transition probability is irrelevant.

Notice, however, that the coherence lengths are infinite because of the infinity of ω, whereas the uncertainty σ_x has to be finite in order to satisfy the condition (3.45) necessary for the coherent production and detection of the different massive neutrino components.

As we have remarked in subsection 2.3 for the similar limit in the case of the production process, the required conditions ($\vec{v}_{D_I} = 0$ and eq. (4.9)) are unlikely to be achieved in any realistic experiment. We think that in the case of the detection process the required conditions are even more unlikely to be achieved than in the case of the production process, because neutrino detection requires in practice to reveal at least one of the final particles in the detection process. This implies that these particles cannot be unlocalized and the condition (4.9) is unrealistic.

4.3 Realistic case

In a realistic case production and detection occur in matter and all the detection particles have uncertainties of the same order of magnitude, as well as all the production particles, as we have discussed in subsection 2.4. Let us consider, for example, the case in which the order of magnitude of the spatial localization of the detection particles is much larger than that of the production particles:

$$\sigma_{xD_I} \simeq \sigma_{xD_F} \simeq \sigma_{x\ell_\beta^-} \simeq 3\sigma_{xD} \gg 3\sigma_{xP} \simeq \sigma_{xP_I} \simeq \sigma_{xP_F} \simeq \sigma_{x\ell_\alpha^+} \,, \tag{4.16}$$

which imply that the momentum uncertainty of the detection process is much smaller than that of the production process, and

$$\vec{v}_D \simeq \frac{1}{3}\left(\vec{v}_{D_I} + \vec{v}_{D_F} + \vec{v}_{\ell_\beta^-}\right) , \qquad \Sigma_D \simeq \frac{1}{3}\left(\vec{v}_{D_I}^{\,2} + \vec{v}_{D_F}^{\,2} + \vec{v}_{\ell_\beta^-}^{\,2}\right) . \tag{4.17}$$

Let us consider, for example, a muon neutrino ($\alpha = \mu$) produced in the pion decay process (2.4) at rest and detected as an electron neutrino ($\beta = e$) in the process

$$\nu_e + {}^{12}\mathrm{C} \to {}^{12}\mathrm{N}_{\mathrm{g.s.}} + e^- \,, \tag{4.18}$$

where ${}^{12}\mathrm{N}_{\mathrm{g.s.}}$ is the ground state of ${}^{12}\mathrm{N}$, with an atomic mass excess $\Delta M({}^{12}\mathrm{N}) \simeq 17.3\,\mathrm{MeV}$. Neglecting the nuclear recoil energy, the energy of the electron is given by

$$E_{e^-} \simeq E - \Delta M({}^{12}\mathrm{N}) \simeq 12.5\mathrm{MeV} \,, \tag{4.19}$$

where E is the energy of a massless neutrino given in eq. (2.71). Let us consider a recoil electron emitted in the backward direction:

$$\vec{v}_{e^-} = -\vec{\ell}|\vec{v}_{e^-}| \,. \tag{4.20}$$

Since $|\vec{v}_{D_F}| \simeq |\vec{v}_{D_I}| = 0$ and $|\vec{v}_{e^-}| \simeq 1$, we have

$$\vec{v}_D \simeq -\frac{1}{3}\vec{\ell}, \qquad \Sigma_D \simeq \frac{1}{3}\,, \qquad \lambda_D \simeq \frac{1}{9}\,. \tag{4.21}$$

From eq. (3.31) we get

$$\rho \simeq \frac{\lambda_D - \vec{\ell}\cdot\vec{v}_D\left(1 - \vec{\ell}\cdot\vec{v}_D\right)}{\lambda_D + \left(\vec{\ell}\cdot\vec{v}_D - 1\right)^2} \simeq 0.29\,. \tag{4.22}$$

This value of ρ is different from the value of ξ in eq. (2.86), albeit of the same order of magnitude. Therefore, the energies and momenta of the massive neutrino contributions to the flavor transition amplitude (3.46) are different from the average energies and momenta of the massive neutrino components of the neutrino state created in the production process.

From eq. (3.50) we obtain

$$\omega \simeq \frac{\lambda_D + (v_D^x - 1)^2 + (v_D^y)^2 + (v_D^z)^2}{\lambda_D + (v_D^y)^2 + (v_D^z)^2} \simeq 17\,, \tag{4.23}$$

and from eq. (4.4) we obtain the coherence lengths

$$L_{ab}^{\mathrm{coh}} \simeq \frac{2\times 10^{16}\mathrm{eV}^2}{|\Delta m_{ab}^2|}\sigma_{xD}\,. \tag{4.24}$$

If $|\Delta m_{ab}^2| \sim 1\mathrm{eV}^2$ and σ_{xD} is given approximately by the inter-atomic distance, $\sigma_{xD} \sim 10^{-8}\,\mathrm{cm}$, we have $L_{ab}^{\mathrm{coh}} \sim 10^3\,\mathrm{km}$, a sufficiently long coherence distance for short-baseline neutrino oscillation experiments.

From eq. (3.36) we find that ζ is approximately given by

$$\zeta \simeq \frac{\lambda_D}{\lambda_D + \left(1 - \vec{\ell}\cdot\vec{v}_D\right)^2} \simeq 0.53\,, \tag{4.25}$$

and the condition (3.45) is satisfied for

$$m_a^2 \ll \left(10^7\mathrm{eV}\right)\sigma_{pD}\,. \tag{4.26}$$

If σ_{xD} is given approximately by the inter-atomic distance, we have $\sigma_{pD} \sim 10^3\,\mathrm{eV}$ and the inequality (2.54) reads $m_a^2 \ll 10^{10}\,\mathrm{eV}^2$, that is certainly satisfied.

5. Conclusions

We have presented a quantum-field-theoretical model of neutrino oscillations in which the neutrino propagating between the production and detection processes is described by the wave-packet state (2.32), which is determined by the production process as naturally expected from causality. Since in real oscillation experiments neutrinos propagate over macroscopically large distances, we think that this model is preferable over the quantum-field-theoretical model of neutrino oscillations with virtual intermediate neutrinos presented in refs. [17, 18].

We have considered production and detection processes of the form (2.3) and (3.1) (other types of production and detection processes can be considered with straightforward changes to the formalism) in which the interacting particles are described by localized wave packets. The gaussian form of the wave packets of interacting particles that we have adopted should be considered as an approximation of the real wave packets. In any case, other wave packets which are sharply peaked around their average momentum lead to the same results, because these results depend on several integrations that are calculated with the saddle-point approximation.

We have obtained the flavor transition probability, presented in eq. (4.2), that is almost identical to the one derived in the quantum-field-theoretical model with virtual intermediate neutrinos [18]. Only the quantity ω, given in eq. (3.50), is somewhat different from the corresponding one in ref. [18].

As already discussed in refs. [18, 26], the flavor transition probability is determined not only by the production process, but also by the detection process [24]. In particular, we have shown that the energies and momenta of the massive neutrino components relevant for the oscillations are in general different from the average energies and momenta of the massive neutrino components of the propagating neutrino state, which are determined only by the production process.

Our result confirms the correctness of the standard expression (4.3) for the oscillation lengths of extremely relativistic neutrinos and the existence of coherence lengths given by eq. (4.4). We agree with the author of ref. [26] on the possibility to extend without limit the coherence length with an appropriate setup of the detection process. We have shown that the coherence length cannot be increased without limit by decreasing the momentum uncertainty of the detection (or production) process, because a vanishing momentum uncertainty does not allow the coherent detection (or production) of different massive neutrino components. Instead, the coherence length can be increased without limit by choosing a detection (or production) process such that ρ, given in eq. (3.31), is zero and ω, given in eq. (3.50), is infinite. As discussed for the example presented in subsection 4.2, in practice the possibility to increase without limit the coherence length is rather unrealistic. As shown in subsection 4.3, in a realistic experimental setup the coherence length is very long, but not infinite.

Finally, let us recall the important remark presented at the end of the introductory section 1. In this paper we have assumed that the particles P_I, P_F and ℓ_α^+ participating to the neutrino production process (2.3) are described by the pure wave-packet states (2.8) in which all their properties are determined. In practice it is common that the knowledge of these properties is less than complete. In this case the particles P_I, P_F and ℓ_α^+ must be described by statistical operators (density matrices) constructed from appropriate incoherent mixtures of the pure wave-packet states (2.8). Consequently, the neutrino created in the production process must be described by a statistical operator constructed from an incoherent mixture of the pure wave-packet states (2.32). Similarly, if there is incomplete knowledge of the properties of the particles P_I, P_F and ℓ_α^+ and the particles D_I, D_F and ℓ_β^- participating, respectively, to the production and detection processes, the oscillation probability is given by an appropriate average of the probability in eq. (4.2) over the unknown quantities.

Acknowledgments

I would like to thank S.M. Bilenky for stimulating discussions and enlightening remarks, and C.W. Kim for a long and fruitful collaboration on the study of the theory of neutrino oscillations. I am specially indebted with M. Beuthe for his remarks concerning the energy uncertainty of wave packets, that helped to correct several wrong equations in the first version of the paper appeared in the electronic archive hep-ph.

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